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A generalized tableau associated with colored convolution trees

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Dedicated to Professor R.G. Stanton on the occasion of his 68th birthday.

Abstract

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This paper considers the tableaux arising from the colors at different levels of arbitrary order convolution trees. The results are related to the elements which constitute the Pascal–Lucas–Turner triangles and to various generalizations of the Fibonacci numbers, some formed by altering the order of the recurrence relations and some by coupling some second order recurrence relations.

1. Introduction

In a previous paper, certain stochastic processes were defined on a sequence of binary trees such that the tree T_n has F_n leaf-nodes where F_n is the nth element of the Fibonacci sequence [16]. In a later paper it was shown how to construct trees so that the nodes were weighted with integers from a general sequence $\{C_n\}$ using a sequential method referred to as the 'dripfeed principle' [12]. Subsequently it was shown how generalized Fibonacci numbers can be used to color convolution trees so that the shades of the trees establish a generalization of Zeckendorf's theorem and its dual [14]. There was also a construction which provided an illustration of the original Zeckendorf theorem, which established the completeness of the Fibonacci sequence and generated the Zeckendorf integer representations.

In the sequence of Fibonacci convolution trees $\{T_n\}$ given in [12], the sum of the weights assigned to the nodes of T_n is equal to the nth term of the convolution of $\{F_n\}$ and $\{C_n\}$. That is, if Ω means the sum of weights, we have

$$\Omega(T_n) = (F * C)_n = \sum_{i=1}^{n} F_i C_{n-i+1}. \tag{1.1}$$

For instance,

$$\begin{aligned}(F*F)_5 &= F_1F_5 + F_2F_4 + F_3F_3 + F_4F_2 + F_5F_1\\ &= 5+3+4+3+5 = 20,\end{aligned}$$

to which we shall refer later.

With the same tree construction, and a modified coloring rule, a graphical 'proof' was given of Zeckendorf's theorem, namely that every positive integer can be represented as the sum of distinct Fibonacci numbers, using no two consecutive Fibonacci numbers, and that such a representation is unique [5].

Given a sequence of colors $C = \{C_1, C_2, C_3, \ldots\}$, we construct kth order colored, rooted trees, T_n, as follows: The first k trees:

$$T_1 = C_1 \bullet; \qquad T_n = T_{n-1} \bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\, C_n, \quad n = 2, 3, \ldots, k,$$

with the root node being C_i for each of these; subsequent trees:

$$T_{n+k} = C_{n+k} \bigvee_{i=0}^{k-1} T_{n+i},$$

using the 'drip-feed' construction, in which the kth order fork operation V is to mount trees $T_n, T_{n+1}, \ldots, T_{n+k-1}$ on separate branches of a new tree with root node colored by C_{n+k}. Thus, for example, when $k=2$, and $C = \{F_n\}$, the sequence of Fibonacci numbers, the first four second-order colored trees are as pictured in Fig. 1.

Now consider the first four trees associated with $F(a, b)$, pictured in Fig. 2. The coloring sequence is the general Fibonacci one, namely, $F(a, b) = \{a, b, a + b, a + 2b, \ldots\}$.

Let (N_a, N_b) represent the number of a's and the number of b's at a given level of a tree. We may tabulate these pairs as in Table 1.

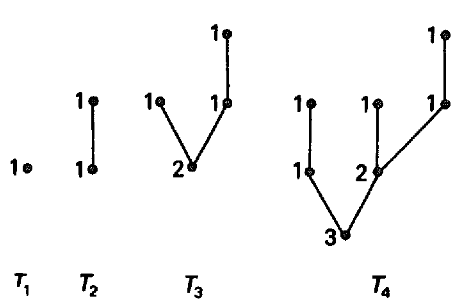

Fig. 1.

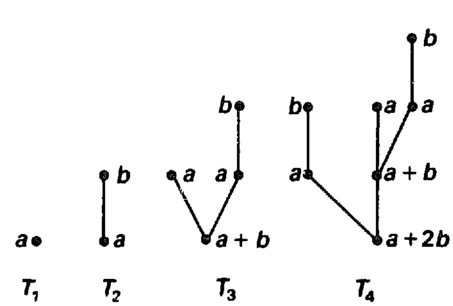

Fig. 2.

Table 1

Level + 1 = m	1	2	3	4	5	6
Tree						
T_1	(1, 0)					
T_2	(1, 0)	(0, 1)				
T_3	(1, 1)	(2, 0)	(0, 1)			
T_4	(1, 2)	(2, 1)	(2, 1)	(0, 1)		
T_5	(2, 3)	(2, 3)	(4, 1)	(2, 2)	(0, 1)	
T_6	(3, 5)	(3, 5)	(4, 4)	(6, 2)	(2, 3)	(0, 1)

If we represent the element in the nth row and mth column of this array by the vector x_{nm}, then x_{nm} satisfies the partial recurrence relation

$$x_{nm} = x_{n-1,m-1} + x_{n-2,m-1}, \quad 1 < m < n, \ n > 2,$$

where the addition of number pairs is elementwise, and the boundary conditions are

$$x_{11} = x_{21} = (1, 0); \qquad x_{n1} = (F_{n-2}, F_{n-1}), \quad n > 2;$$

$$x_{22} = (0, 1); \qquad x_{nm} = (0, 0), \quad m > n.$$

It is the main purpose of this paper to generalize this result.

2. Generalized tableau

The tableau can be generalized for arbitrary k as follows: Consider x_{nm} as a k-component vector, with x_{nm} equal to the null vector when $m > n$, and

$$x_{nm} = \sum_{i=1}^{k} x_{n-i,m-1}, \quad 1 < m < n, \ n > k, \tag{2.1}$$

with

$$x_{n1} = (U_{1n}, U_{2n}, \ldots, U_{kn}), \quad n = 1, 2, \ldots, k,$$

where $\{U_{sn}\}$, $s = 1, 2, \ldots, k$, are the k 'basic' sequences of order k defined by the recurrence relation

$$U_{sn} = \sum_{j=1}^{k} U_{s,n-j}, \quad n > k \tag{2.2}$$

with initial terms when $n = 1, 2, \ldots, k$, $U_{sn} = \delta_{sn}$, (Shannon and Bernstein [10]), where δ_{ij} is the Kronecker delta.

When $k = 2$, we have, as before, that if we represent the element in the nth row and mth column of this array by x_{nm}, then x_{nm} satisfies the partial recurrence

Table 2

n	1	2	3	4	5	6	7	8	9
U_{1n}	1	0	0	1	1	2	4	7	13
U_{2n}	0	1	0	1	2	3	6	11	20
U_{3n}	0	0	1	1	2	4	7	13	24

relation

$$x_{nm} = x_{n-1,m-1} + x_{n-2,m-1}, \quad 1 < m < n,\ n > 2,$$
$$x_{nm} = (\delta_{1m}, \delta_{2m}), \quad n = 1, 2;\ 1 \leqslant m \leqslant n$$

with boundary conditions $x_{n1} = (F_{n-2}, F_{n-1})$ and $x_{nm} = (0, 1)$.

As illustrations of the $\{U_{sn}\}$ we have Table 2 when $k = 3$ and Table 3 when $k = 4$.

Various properties of $\{U_{sn}\}$ have been developed by Shannon [9]. To see more easily what follows, it is useful to continue the tree table of (N_a, N_b) for $k = 2$ (see Table 4).

It can be observed in Tables 4 and 5 that, for $n > k$ and $m > 1$,

$$x_{nm} = \sum_{i=1}^{k} x_{n-i,m-1}. \tag{2.3}$$

As explained elsewhere [17], the rule of formation comes directly from the construction of the trees. When $k = 3$, we have the array as shown in Table 5.

The first main result is that for $m = 1, 2, \ldots, [(n-1)/k]$, ([ ] is the greatest integer function),

$$x_{nm} = x_{n1}.$$

Proof. The proof follows from induction on m by utilizing the results

$$\begin{aligned} x_{n2} &= \sum_{i=1}^{k} x_{n-i,1} = \sum_{i=1}^{k} (U_{1,n-i}, U_{2,n-i}, \ldots, U_{k,n-i}) \\ &= \left(\sum_{i=1}^{k} U_{1,n-i} \sum_{i=1}^{k} U_{2,n-i}, \ldots, \sum_{i=1}^{k} U_{k,n-i}\right) = (U_{1,n}, U_{2,n}, \ldots, U_{k,n}) \\ &= x_{n1}, \text{ and so on.} \quad \square \end{aligned}$$

Table 3

n	1	2	3	4	5	6	7	8	9
U_{1n}	1	0	0	0	1	1	2	4	8
U_{2n}	0	1	0	0	1	2	3	6	12
U_{3n}	0	0	1	0	1	2	4	7	14
U_{4n}	0	0	0	1	1	2	4	8	15

Table 4

m	1	2	3	4	5	
T_7	(5, 8)	(5, 8)	(5, 8)	(8, 5)	(8, 4)	
T_8	(8, 13)	(8, 13)	(8, 13)	(9, 12)	(14, 7)	
T_9	(13, 21)	(13, 21)	(13, 21)	(13, 21)	(17, 17)	
T_{10}	(21, 34)	(21, 34)	(21, 34)	(21, 34)	(22, 33)	
T_{11}	(34, 55)	(34, 55)	(34, 55)	(34, 55)	(34, 55)	
m	6	7	8	9	10	11
T_7	(2, 4)	(0, 1)				
T_8	(10, 7)	(2, 5)	(0, 1)			
T_9	(22, 11)	(12, 11)	(2, 6)	(0, 1)		
T_{10}	(31, 24)	(32, 18)	(14, 16)	(2, 7)	(0, 1)	
T_{11}	(29, 50)	(53, 35)	(44, 29)	(16, 22)	(2, 8)	(0, 1)

For instance, when $k=3$,

$$x_{72}=(4,6,7)=(U_{17},U_{27},U_{37}),$$
$$x_{10,3}=(24,37,44)=(U_{1,10},U_{2,10},U_{3,10});$$

when $k=2$,

$$x_{52}=(2,3)=(U_{15},U_{25}),$$
$$x_{73}=(5,8)=(U_{17},U_{27}),$$
$$x_{94}=(13,21)=(U_{19},U_{29}),$$
$$x_{11,5}=(34,55)=(U_{1,11},U_{2,11}),$$

in which $U_{1n}=U_{2,n-1}=F_{n-2}$ in the conventional Fibonacci notation.

Table 5

m	1	2	3	4	5
T_1	(1, 0, 0)				
T_2	(1, 0, 0)	(0, 1, 0)			
T_3	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)		
T_4	(1, 1, 1)	(3, 0, 0)	(0, 2, 0)	(0, 0, 1)	
T_5	(1, 2, 2)	(3, 1, 1)	(3, 2, 0)	(0, 2, 1)	(0, 0, 1)
T_6	(2, 3, 4)	(3, 3, 3)	(6, 2, 1)	(3, 4, 1)	(0, 2, 2)
T_7	(4, 6, 7)	(4, 6, 7)	(9, 4, 4)	(9, 6, 1)	(3, 6, 3)
T_8	(7, 11, 13)	(7, 11, 13)	(10, 10, 11)	(18, 8, 5)	(12, 12, 3)
T_9	(13, 20, 24)	(13, 20, 24)	(14, 20, 23)	(25, 16, 16)	(20, 18, 7)
T_{10}	(24, 37, 44)	(24, 37, 44)	(24, 37, 44)	(33, 34, 38)	(52, 30, 22)
m	6	7	8	9	10
T_6	(0, 0, 1)				
T_7	(0, 2, 3)	(0, 0, 1)			
T_8	(3, 8, 6)	(0, 2, 4)	(0, 0, 1)		
T_9	(15, 20, 8)	(3, 10, 10)	(0, 2, 5)	(0, 0, 1)	
T_{10}	(35, 36, 13)	(18, 30, 17)	(3, 12, 15)	(0, 2, 6)	(0, 0, 1)

The second result is that for $m > [(n-1)/k]$, x_{nm} is formed from the boundary conditions

$$x_{k+1,m} = (0, 0, \ldots, k-m+2, \ldots, 0)$$

in which the nonzero position is the $(m-1)$th; thereafter, the elements are generated by the algorithm defined by the vector difference operator Δ, such that if

$$\Delta x_{nm} = x_{n+1,m+1} - x_{n,m},$$

then the sth order difference is given by

$$\Delta^s x_{n+s,n} = (0, 0, \ldots, 0, 1), \quad \text{for } n \geq k.$$

The proof follows from the initial conditions and the ordinary recurrence relation (2.2) for $\{U_{sn}\}$ to get $x_{k+1,m}$, and then from the partial recurrence relation (2.3) for $x_{k+n,k+n-i}$.

As examples, we have when $k = 2$,

$$\begin{array}{lll}
x_{42} = (2, 1) & & \\
 & \Delta x_{42} = x_{53} - x_{42} = (2, 0) & \\
x_{53} = (4, 1) & & \Delta^2 x_{42} = (0, 1) \\
 & \Delta x_{53} = x_{64} - x_{53} = (2, 1) & \\
x_{64} = (6, 2) & & \Delta^2 x_{53} = (0, 1) \\
 & \Delta x_{64} = x_{75} - x_{64} = (2, 2) & \\
x_{75} = (8, 4) & & \Delta^2 x_{64} = (0, 1) \\
 & \Delta x_{75} = x_{86} - x_{75} = (2, 3) & \\
x_{86} = (10, 7) & & \Delta^2 x_{75} = (0, 1) \\
 & \Delta x_{86} = x_{97} - x_{86} = (2, 4) & \\
x_{97} = (12, 11) & & \Delta^2 x_{86} = (0, 1) \\
 & \Delta x_{97} = x_{10,8} - x_{97} = (2, 5) & \\
x_{10,8} = (14, 16) & & \Delta^2 x_{97} = (0, 1) \\
 & \Delta x_{10,8} = x_{11,9} - x_{10,8} = (2, 6) & \\
x_{11,9} = (16, 22) & &
\end{array}$$

When $k = 4$

$$\begin{array}{lll}
x_{52} = (4, 0, 0, 0) & & \\
 & \Delta x_{52} = (0, 3, 0, 0) & \\
x_{63} = (4, 3, 0, 0) & & \Delta^2 x_{52} = (0, 0, 2, 0) \\
 & \Delta x_{63} = (0, 3, 2, 0) & \\
x_{74} = (4, 6, 2, 0) & & \Delta^2 x_{63} = (0, 0, 2, 1) \\
 & \Delta x_{74} = (0, 3, 4, 1) &
\end{array}$$

$$
\begin{array}{lll}
x_{85} = (4, 9, 6, 1) & & \Delta^2 x_{74} = (0, 0, 2, 2) \\
& \Delta x_{85} = (0, 3, 6, 3) & \\
x_{96} = (4, 12, 12, 4) & & \Delta^2 x_{85} = (0, 0, 2, 3) \\
& \Delta x_{96} = (0, 3, 8, 6) & \\
x_{10,7} = (4, 15, 20, 10) & &
\end{array}
$$

and

$$
\begin{array}{lll}
x_{53} = (0, 3, 0, 0) & & \\
& \Delta x_{53} = (0, 0, 2, 0) & \\
x_{64} = (0, 3, 2, 0) & & \Delta^2 x_{53} = (0, 0, 0, 1) \\
& \Delta x_{64} = (0, 0, 2, 1) & \\
x_{75}(0, 3, 4, 1) & & \Delta^2 x_{64} = (0, 0, 0, 1) \\
& \Delta x_{75} = (0, 0, 2, 2) & \\
x_{86} = (0, 3, 6, 3) & & \Delta^2 x_{75} = (0, 0, 0, 1) \\
& \Delta x_{86} = (0, 0, 2, 3) & \\
x_{97} = (0, 3, 8, 6) & &
\end{array}
$$

and

$$
\begin{array}{ll}
x_{54} = (0, 0, 2, 0) & \\
& \Delta x_{54} = (0, 0, 0, 1) \\
x_{65} = (0, 0, 2, 1) & \\
& \Delta x_{65} = (0, 0, 0, 1) \\
x_{76} = (0, 0, 2, 2) & \\
& \Delta x_{76} = (0, 0, 0, 1) \\
x_{87} = (0, 0, 2, 4). &
\end{array}
$$

3. Connections with Pascal-T triangles

Turner [14–15] has defined the level counting function

$$L = \binom{n}{m \mid i}$$

as the number of nodes in T_n which at level m are colored C_i, where T_n is the tree colored by integers of the sequence $C = \{C_1, C_2, C_3, \ldots\}$.

One of the results proved is that

$$\binom{n}{m \mid 1} = \sum_{j=1}^{k} \binom{n-j}{m-1 \mid 1}. \tag{3.1}$$

It is also shown in effect that

$$U_{k,k+n} = \sum_{m=0}^{n} \binom{n}{m \mid 1}.$$

Thus

$$U_{k,k+n}=\sum_{m=0}^{n}\sum_{j=1}^{k}\binom{n-j}{m-1\,|\,1}. \tag{3.2}$$

It is also worth noting that (3.1) has the same form as (2.2). Now $U_{k,k+n}$ is, in the terminology of Macmahon [7], the homogeneous product sum of weight n of the zeros α_j, $j=1,2,\ldots,k$, assumed distinct, of the auxiliary polynomial, $f(x)$, associated with the linear recurrence relation for $\{U_{k,k+n}\}$. Shannon and Horadam [11] have proved that formally

$$\sum_{n=1}^{\infty} U_{k,k+n}x^n=\left(x^k f\left(\frac{1}{x}\right)\right)^{-1}.$$

Thus if we expand the right-hand side of

$$\sum_{n=1}^{\infty} U_{k,k+n}x^n=1/(1-x-x^2-\cdots-x^k)$$

by the multinomial theorem and equate corresponding coefficients of powers of x we get

$$U_{k,k+n}=\sum_{\Sigma_i\lambda_i=n}\frac{(\Sigma\,\lambda_i)!}{\lambda_1!\,\lambda_2!\cdots\lambda_k!} \tag{3.3}$$

which agrees with the analogous result in Macmahon. This is worth noting because Turner [15] has shown that the $\binom{m}{n\,|\,1}$ are multinomial coefficients generated from $x(x+x^2+x^3+\cdots+x^k)^m$. For example,

$$U_{2,2+n}=\sum_{\Sigma_i\lambda_i=n}\frac{(\Sigma\,\lambda i)!}{\lambda_1!\,\lambda_2!}=\sum_{s+2m=n}\binom{n-m}{m}$$

where $\lambda_1=s$ and $\lambda_2=m$, as in Barakat [4]; and

$$\begin{aligned}U_{3,3+n}&=\sum_{\Sigma_i\lambda_i=n}\frac{(\Sigma\,\lambda_i)!}{\lambda_1!\,\lambda_2!\,\lambda_3!}=\sum_{s+2m+3t=n}\frac{(n-m-2t)}{s!\,m!\,t!}\\&=\sum_{s+2m+3t=n}\binom{n-m-2t}{m+t}\binom{m+t}{t}\end{aligned}$$

where $\lambda_1=s$, $\lambda_2=m$ and $\lambda_3=t$, as in Shannon [8].

4. Other generalizations

We can also develop trees for other generalizations. For instance, Atanassov [1, 3] defines 2-F-sequences

$$\alpha_{n+2}=\beta_{n+1}+\beta_n,\qquad \beta_{n+2}=\alpha_{n+1}+\alpha_n,\qquad n\geqslant 0 \tag{4.1}$$

with $\alpha_0=a$, $\alpha_1=b$, $\beta_0=c$, $\beta_1=d$ fixed real numbers. The trees for this scheme are shown in Fig. 3.

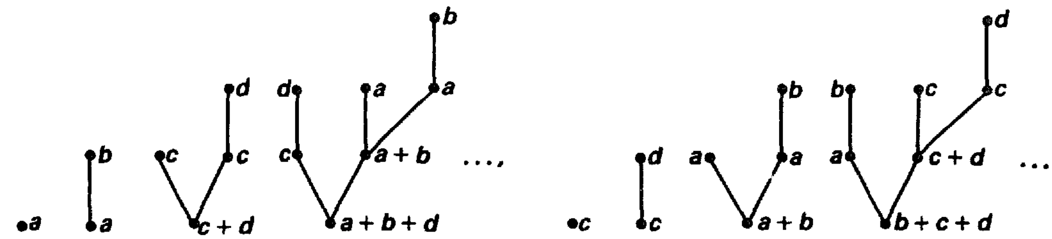

Fig. 3.

Similarly, there are 7 basic 3-F-sequences (Atanassov [2]), two of which are shown with their corresponding sets of trees in Fig. 4.

$$\begin{aligned} \alpha_{n+2} &= \gamma_{n+1} + \gamma_n, \\ \beta_{n+2} &= \alpha_{n+1} + \alpha_n, \\ \gamma_{n+2} &= \beta_{n+1} + \beta_n. \end{aligned} \tag{4.2a}$$

$$\begin{aligned} \alpha_{n+2} &= \beta_{n+1} + \gamma_n, \\ \beta_{n+2} &= \alpha_{n+1} + \alpha_n, \\ \gamma_{n+2} &= \gamma_{n+1} + \beta_n. \end{aligned} \tag{4.2b}$$

These trees all have the same structure as the Fibonacci convolution trees, but their node colorings are different since their coloring rules are determined by coupled recurrences such as those of (4.1) and (4.2).

One simple illustration of how studies of the colors arising on the trees lead to interesting tableaux with Fibonacci properties is the following: For the two tree sequences S_1 and S_2 (say) from the 2-F scheme, we may compute the total weight (i.e. sum of the node colors) for each tree. For example, the fourth tree in sequence S_1 has weight $4a + 3b + 1c + 2d$. Then we may tabulate the coefficients of a, b, c, d, for each sequence (up to the seventh tree as in Tables 6 and 7.

As we expect from the manner in which the trees were colored (following (4.1)), the table for S_2 is table S_1 with its columns permuted thus: $(a, c)(b, d)$. Note that the sequence of row sums is $\{1, 2, 5, 10, \ldots\} = (F * F)$, the convolution of the Fibonacci sequence with itself as in (1.1).

If we add Table 6 and Table 7, elementwise, we get Table 8.

We see that the sum of the weights of the nth trees from the two sequences is:

$$W_n^{(1)} + W_n^{(2)} = U_n(a + c) + V_n(b + d),$$

where

$$\{U_n\} = 1, 1, 3, 5, 10, 18, 33, \ldots \quad \text{and} \quad \{V_n\} = 0, 1, 2, 5, 10, 20, 38, \ldots.$$

Now $V_n = (F * F)_{n-1}$ (proof given below); and $U_n + V_n = (F * F)_n$ (since Σ_n in the table is $2(F * F)_n$), therefore

$$U_n = (F * F)_n - (F * F)_{n-1}.$$

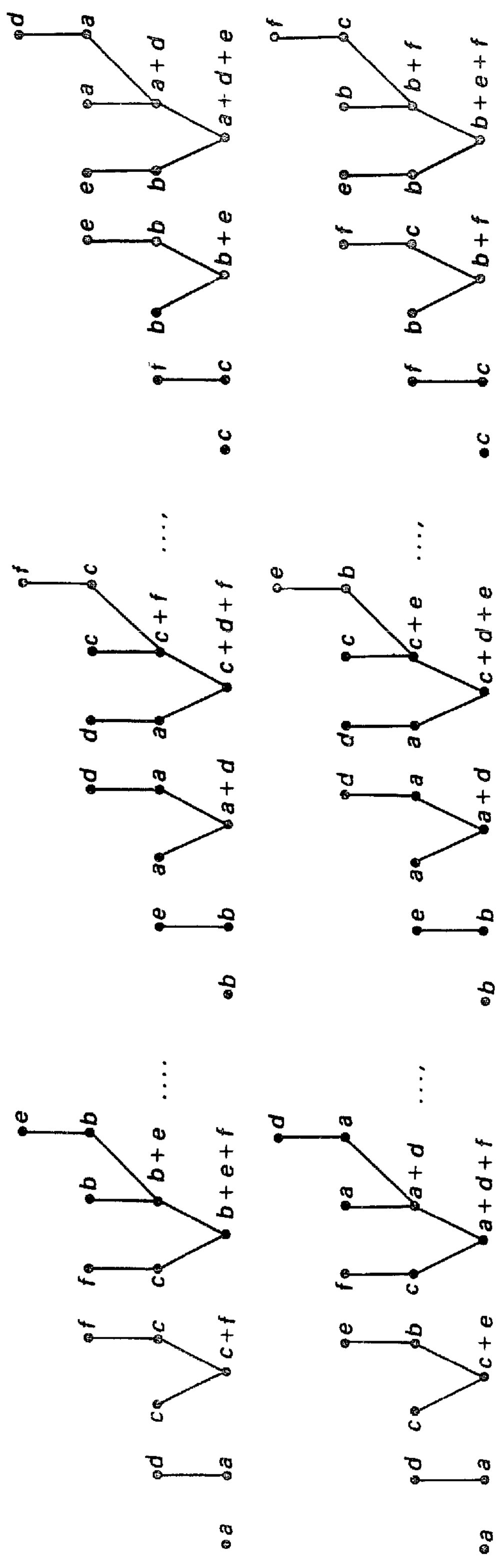

Fig. 4.

Table 6

Tree	S_1 Coefficients of a	b	c	d	Σ
T_1	1	0	0	0	1
T_2	1	1	0	0	2
T_3	0	0	3	2	5
T_4	4	3	1	2	10
T_5	5	6	5	4	20
T_6	7	8	11	12	38
T_7	19	20	14	18	71

In [6] the following identity for the Fibonacci convolution term is given

$$5(F*F)_{n-1} = (n+1)F_{n-1} + (n-1)F_{n+1}.$$

Using this we obtain

$$V_n = \tfrac{1}{5}[(n+1)F_{n-1} + (n-1)F_{n+1}];$$

and

$$\begin{aligned} 5U_n &= [(n+2)F_n + nF_{n+2}] - [(n+1)F_{n-1} + (n-1)F_{n+1}] \\ &= (n+1)(F_n + F_{n-2}) + F_{n+1} \end{aligned}$$

therefore $U_n = \frac{1}{5}[(n+1)L_{n-1} + F_{n+1}]$, where L_{n-1} is a Lucas number.

We finally prove the convolution forms given above for U_n, V_n.

Proof. We showed in [12] that if a single sequence of the convolution trees is colored sequentially, using color C_n of a sequence $\{C_n\}$ to color the root node of T_n, and mounting the previously colored T_{n-1} and T_{n-2} on the fork, then the weight of T_n is $(F*C)_n$.

Now the general term of Σ_n (in the above table) is obtained by setting $a=b=c=d=1$; in that event, both S_1 and S_2 are Fibonacci convolution trees (i.e., $C=F$ in both cases), so $\Sigma_n = 2(F*C)_n$.

Similarly, if we set $a=0=c$ and $b=1=d$, we find that S_1 and S_2 are identical but with color sequences $\{F_{n-1}\}$; and then $U_n \cdot 0 + V_n \cdot 2 = 2(F*F)_{n-1}$, giving the required form of V_n.

Table 7

Tree	S_2 Coefficients of a	b	c	d	Σ
T_1	0	0	1	0	1
T_2	0	0	1	1	2
T_3	3	2	0	0	5
T_4	1	2	4	3	10
T_5	5	4	5	6	20
T_6	11	12	7	8	38
T_7	14	18	19	20	71

Table 8
Total weights $W_n^{(1)} + W_n^{(2)}$, where $W_n^{(i)}$ is the weight of T_n in sequence S_i

n	a	b	c	d	Σ_n
1	1	0	1	0	2
2	1	1	1	1	4
3	3	2	3	2	10
4	5	5	5	5	20
5	10	10	10	10	40
6	18	20	18	20	76
7	33	38	33	38	142

Acknowledgement

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